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COMMENTS ON GOOD'S PROPOSAL FOR NEW RULES OF QUANTIZATION

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Abstract

In a recent paper [1] Good postulated new rules of quantization, one of the major features of which is that the quantum evolution of the wave function is always given by ordinary differential equations. In this paper we analyse the proposal in some detail and discuss its viability and its relationship with the standard quantum theory. As a byproduct, a simple derivation of the 'mass spectrum' for the Klein-Gordon field is presented, but it is also shown that there is a complete additional spectrum of negative 'masses'. Finally, two major reasons are presented against the viability of this alternative proposal:

a) It does not lead to the correct energy spectrum for the hydrogen atom.

b) For field models, the standard quantum theory cannot be recovered from this alternative description.

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I. Introduction

In a recent paper [1] (see also [2]) Good has proposed new rules for (second) quantizing field models. The necessary postulates of interpretation for this new formalism are missing in the paper of Good, but natural ones can be found (see below). The quantization rules proposed by Good, together with these natural rules of interpretation, constitute a quantum theory, different from the usual or standard one, which we will refer to as 'new' or 'finite' quantum theory.

The 'new' quantum theory proposed by Good possesses a number of remarkable properties which deserve to pay attention to it:

- The rules of quantization treat time and spatial coordinates on the same footing and are explicitly covariant, giving rise to a quantum theory that is always explicitly Poincaré invariant. This therefore avoids the proofs of consistency which are necessary in the standard approach.

- Everything in the quantum theory - equations of motions, integration measures, etc. - is based on finite-dimensional calculus, thereby avoiding the problems of infinities that appear in the standard approach.

Moreover, this formalism would lead to a discrete 'spectrum' of allowed masses for the particles of the quantum theory.

The purpose of this paper is to study what lies 'behind' all these interesting properties and to check whether or not this alternative proposal has any chance of competing with the standard one.

Our conclusion is that in spite of all the advantages of this proposal, it does not reproduce the experimental data and therefore is *not* an acceptable alternative to the standard one.

The paper is organized as follows: In Section 2 we analyse the Good's quantization rules in some detail and put them in the context of the finite-

dimensional covariant approach to field theory. In Section 3, we develop several topics of the new theory for several elementary systems and check its experimental predictions. We also re-derive the 'mass' spectrum for the Klein-Gordon field. Section 4 is devoted to a comparative description of the interpretation rules of this theory in relation to the standard one.

II. The hard way to a finite-dimensional quantum field theory and Good's proposal

There are bassicaly two different ways of looking at field theory from the standpoint of classical mechanics, or, in other words, there are two different ways of considering field theory as a generalization of classical mechanics. These can be summarized as follow:

	Classical mechanics.	**Classical field theory.**
A.	$q_n(t)$: n discrete label.	$\varphi^a(\mathbf{x})(t)$: $\mathbf{x}$ continuous label.
B.	$q_n(t)$: Space-time $(1+0)-$ dimensional.	$\varphi^a(\mathbf{x},t)$: Space-time $(1+n)-$ dimensional.

In the perspective A the solutions of the classical equations of motion are sections of an infinite-dimensional fiber bundle with co-ordinates $(\varphi^a(\mathbf{x}),t)$ - and base manifold co-ordinatized by t-, whereas in the perspective B the solutions of the equations of motion are sections of a finite-dimensional fiber bundle co-ordinatized by $(\varphi^a,\mathbf{x},t)$ - and base manifold co-ordinatized by $(\mathbf{x},t)$ -. Although both of these points of view lead to the same classical field theories, this is not the case for the quantum ones: these different interpretations lead to different quantum theories. The interpretation A leads to the standard quantum field theory, whereas the interpretation B

would lead to a *different* quantum field theory, if any were ever constructed.

Quantum Mechanics is described by wave functions $\Psi(\mathbf{q},t)$, the interpretation of which is:

$|\Psi(\mathbf{q}_0,t)|^2$ *is the probability of finding the result* $\mathbf{q}_0$ *if a measure of* $\mathbf{q}$ *is made at the time* t*.*

This interpretation of Quantum Mechanics, together with the two ways, summarized above, of considering Field Theory as a generalization of Mechanics, would result in the two different descriptions of the quantum theory that follow:

A1. The quantum theory is described by wave functions (functionals) $\Psi(\{\varphi(\mathbf{x})\},t)$ the interpretation of which is:

$|\Psi(\{\varphi^a(\mathbf{x})\}_0,t)|^2$ *is the probability of finding the result* $\{\varphi^a(\mathbf{x})\}_0$ *if a measure of the configuration of the field is made at the time* t*.*

[This is the standard description of Quantum Field Theory.]

B1. The quantum theory is described by wave functions $\Psi(\varphi^a,\mathbf{x},t)$ the interpretation of which is:

$|\Psi(\varphi^a_0,\mathbf{x},t)|^2$ *is the probability that a measure of the field* φ *at the point* $(\mathbf{x},t)$ *of the space-time gives the result* φ^a_0*.*

There is no *a priory* reason for the correct description of the quantum theory to be the one in A1 and not that in B1. In fact, the description in B1 seems to be better than that in A1 because

- It treat space co-ordinates and time on the same footing thus providing a more suitable framework to construct, throught covariant rules of quantization, a covariant quantum theory.

- The wave functions of the theory are proper functions, not functionals, avoiding, from the very beginning, the problems of infinities of the standard picture.

- The basic questions which it would be suitable to answer are of a more local nature than those of the standard approach.

However, a (successful) quantum theory based on the point of view A is known (the standard one) whereas all attempts to construct a quantum theory making use of the point of view in B have failed (Recent discussions on this subject can be found, for instance, in Ref. [3] and references therein). In fact it is easy to show (see below, Section 4) that the descriptions B1 and A1 are *not* equivalent to each other and, therefore, if the description in A1 is considered as the right one, then the description in B1 cannot be correct as well.

Apart from the problems of interpretation, there are additional obstacles in the construction of a formalism for this alternative quantum theory. The main problem in this approach appears to be that there is no natural notion of Poisson bracket. Hence, in this covariant formalism, there is no natural way of obtaining quantization rules with which to construct the quantum theory from the classical one. There is a covariant generalization of the Legendre transform of Mechanics as well as a covariant Hamiltonian, but the covariant Hamiltonian equations of motion cannot be obtained by means of a Poisson bracket.

Given a Lagrangian $\mathcal{L} = \mathcal{L}(\phi^a, \partial_\mu \phi^a)$, the covariant Hamiltonian $\mathcal{H}$ is obtained by means of the covariant legendre transform:

$$\mathcal{H} = \pi_a^\mu \partial_\mu \phi^a - \mathcal{L} \tag{1}$$

where the covariant momenta π_a^μ are defined by:

$$\pi_a^\mu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi^a\right)} \tag{2}$$

If we now write the Lagrangian in the following covariant Hamiltonian form:

$$\mathcal{L} = \pi_a^\mu \partial_\mu \phi^a - \mathcal{H}(\phi^a, \pi_a^\mu) \tag{3}$$

its Lagrange equations of motion will also have a covariant Hamiltonian form:

$$\partial_\mu \phi^a = \frac{\partial \mathcal{H}}{\partial \pi_a^\mu} \tag{4}$$

$$\partial_\mu \pi_a^\mu = -\frac{\partial \mathcal{H}}{\partial \phi^a} \tag{5}$$

The obstacle now is that these equations of motion cannot be associated with a pair {Poisson bracket, Hamiltonian}. Therefore, the basic tool in the standard formalism for constructing the quantum theory from the classical one is lacking here.

In the light of the above discussion, let us consider now the proposal of Good [1, 2].

The quantization rules are:

I. For a field model defined in a $(1+n)$-dimensional space-time coordinatized by $(\mathbf{x}, t)$ and with fields φ^a, the quantum theory is described by wave functions $\Psi(\varphi^a, \mathbf{x}, t)$.

[Therefore, the description of the quantum theory is the one in B1.]

II. The correspondence principle is:

$$\pi_a^\mu \Longrightarrow \widehat{\pi}_a^\mu = -\hbar^2 \frac{\partial^2}{\partial \varphi^a \partial x_\mu} \tag{6}$$

III. The quantum equation of motion, the analog of the Schrödinger equation, is

$$\mathcal{H}(\phi^a, -\frac{\partial^2}{\partial\varphi^a \partial x_\mu})\Psi(\varphi^a, x^\beta) = -\hbar^2 \frac{\partial^2}{\partial x^\nu \partial x_\nu}\Psi(\varphi^a, x^\beta) \tag{7}$$

A number of features of these quantization rules are inmediately noticeable:

- The new quantization rules *do not* lead to the usual quantization rules in the mechanical case - when the 'space-time' is $(1+0)$-dimensional.

Hence, they should not be looked at as a generalization for field theory of the familiar quantization rules for Mechanics; they are instead new quantization rules that serve for Mechanics as well as for Field Theory. The quantum theory of Mechanics which will be obtained from these new rules of quantization *is not* the usual Quantum Mechanics. Therefore, since the standard quantization rules have proved extraordinarily successful in predicting all the known experimental data, the next step should be to check whether or not this new quantum theory leads to the same predictions as the standard one. We shall return to this point in Section 3.

- The formula (6) and eq. (2) implies the following equations for the dimensionalities of the quantities involved [In the sequel we shall make $c = 1$, $[c] = 0$; and all dimensionalities will be expressed in terms of $[x]$ and $[m]$.]:

$$[\mathcal{L}] + [x] - [\varphi^a] = [\pi_a^\mu] = 2[\hbar] - [\phi^a] - [x] \Rightarrow [\mathcal{L}] = 2[\hbar] - 2[x] \tag{8}$$

This dimensionality for $\mathcal{L}$ is, in general, different from that required for the standard formalism where $[\mathcal{L}] = [\hbar] - (1+n)[x]$. Ref. [1] addresses this problem simply by arguing that the dimensionality of the Lagrangian is relevant only for the quantum theory - the standard or the new one - and that the only relevant point here is that the new approach requires dimensionalities for the physical quantities different from those in the standard

approach. Another solution for obtaining the correct dimensionalities might be to multiply the Lagrangian for an adequate factor.

Although these positions could be self-consistent in some cases, there is a different point of view that is equivalent to the second above and, furthermore, is in greater agreement with the historical development of (standard) Quantum Mechanics. The key point is that the standard quantization rules required the introduction of a parameter $\hbar$ with dimensionalities $[\hbar] = [m] + [x]$. Hence, we should say that the new rules of quantization also require the introduction of a new parameter with the adequate dimensionalities. Moreover, since there are no physical reasons for the right-hand side of the Schrödinger-like equation to be exactly the form above, we can- and we shall -modify and generalize the second and third rules of quantization by introducing two parameters λ and β in the following way:

II'. The correspondence principle is:

$$\pi_a^\mu \Longrightarrow \widehat{\pi}_a^\mu = -\frac{\hbar^2}{\lambda}\frac{\partial^2}{\partial\varphi^a \partial x_\mu} , \quad [\lambda] = 2[m] - [\mathcal{L}] \tag{9}$$

III'. The quantum equation of motion, the analogue of the Schrödinger equation, is

$$\mathcal{H}(\phi^a, -\frac{\hbar^2}{\lambda}\frac{\partial^2}{\partial\varphi^a \partial x_\mu})\Psi(\varphi^a, x^\beta) = -\frac{\hbar^2}{\lambda\beta}\frac{\partial^2}{\partial x^\nu \partial x_\nu}\Psi(\varphi^a, x^\beta) , \quad [\beta] = 0 \tag{10}$$

III. The new quantum theory for the harmonic oscillator, the Klein-Gordon field and the hydrogen atom

In this section, we shall check the experimental predictions of this new

quantum theory. We shall show that it reproduces the standard energy spectrum for the harmonic oscillator surprisingly well, but that it does not, however, reproduce the right spectrum for the hydrogen atom. The 'mass' spectrum for the Klein-Gordon field is also found in a very simple manner.

III.I. The harmonic oscillator

The familiar Lagrangian for the harmonic oscillator is:

$$\mathcal{L}_{H.O.} = \frac{1}{2}m\left[\dot{\mathbf{q}}^2 - \omega^2\mathbf{q}^2\right] \; ; \quad \mathbf{q} = (q_1, .., q_D) \tag{11}$$

It has dimension $[\mathcal{L}] = [m]$.

The Hamiltonian can be written as

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + m\omega^2\frac{\mathbf{q}^2}{2} \tag{12}$$

The Schrödinger-like equation takes the form:

$$\left[\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\mathbf{q}^2}\right)\left(-\frac{\hbar^2}{\lambda^2}\frac{\partial^2}{\partial t^2}\right) + m\omega^2\frac{\mathbf{q}^2}{2}\right]\Psi(\mathbf{q},t) = -\frac{\hbar^2}{\lambda\beta}\frac{\partial^2}{\partial t^2}\Psi(\mathbf{q},t) \tag{13}$$

Stationary wave functions of definite energy E will satisfy:

$$-\hbar^2\frac{\partial^2}{\partial t^2}\Psi(\mathbf{q},t) = E^2\Psi(\mathbf{q},t) \tag{14}$$

For these functions the (stationary) Schrödinger-like equation is:

$$\left[-\frac{\hbar^2}{2\left\{\frac{m\lambda^2}{E^2}\right\}}\frac{\partial^2}{\partial\mathbf{q}^2} + \left\{\frac{m\lambda^2}{E^2}\right\}\frac{E^2}{\lambda^2}\omega^2\frac{\mathbf{q}^2}{2}\right]\Psi(\mathbf{q},t) = \frac{E^2}{\lambda\beta}\Psi(\mathbf{q},t) \tag{15}$$

It is clear from this equation that the allowed energies will be the solutions of the equation

$$\frac{E^2}{\lambda\beta} = \hbar\sqrt{\frac{E^2}{\lambda^2}\omega^2}\left(n + \frac{D}{2}\right), \qquad n \in \mathcal{N} \tag{16}$$

The energy levels of the new harmonic oscillator are therefore:

$$E = \pm\beta\hbar\omega\left(n + \frac{D}{2}\right), \qquad n \in \mathcal{N} \tag{17}$$

Except for the duplication in positive and negative energies, making $\beta = 1$, eq. (17) exactly reproduces the familiar energy levels of the standard approach.

III.II. The mass spectrum of the Klein-Gordon field

The discussion in the preceding subsection makes it very simple to get the mass spectrum for the Klein-Gordon field already found, albeit in a rather cumbersome manner, in Ref. [1]

The Lagrangian for the Klein-Gordon field is:

$$\mathcal{L} = \frac{1}{2}\hbar^2\left(\partial_\mu\phi_1\partial_\mu\phi_1 + \partial_\mu\phi_2\partial_\mu\phi_2\right) - \frac{1}{2}m_0^2\left(\phi_1^2 + \phi_2^2\right) . \tag{18}$$

The Schrödinger-like equation takes the form:

$$\begin{aligned}\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial\phi_1^2} + \frac{\partial^2}{\partial\phi_2^2}\right)(-\frac{\hbar^2}{\lambda^2}\frac{\partial^2}{\partial x^\nu\partial x_\nu}) + \frac{1}{2}m_0^2\left(\phi_1^2 + \phi_2^2\right)\right]\Psi(\phi_1, \phi_2, x^\mu)&\\ = -\frac{\hbar^2}{\lambda\beta}\frac{\partial^2}{\partial x^\nu\partial x_\nu}\Psi(\phi_1, \phi_2, x^\mu)&\end{aligned} \tag{19}$$

For stationary wave functions:

$$-\hbar^2\frac{\partial^2}{\partial x^\nu\partial x_\nu}\Psi(\phi_1, \phi_2, x^\mu) = m^2\Psi(\phi_1, \phi_2, x^\mu) \tag{20}$$

it can be written in the convenient form:

$$\left[-\frac{\hbar^2}{2\hbar^2\lambda^2/m^2}\left(\frac{\partial^2}{\partial\phi_1^2} + \frac{\partial^2}{\partial\phi_2^2}\right) + \left\{\frac{\hbar^2\lambda^2}{2m^2}\right\}\frac{m_0^2m^2}{\hbar^2\lambda^2}\left(\phi_1^2 + \phi_2^2\right)\right]\Psi(\phi_1, \phi_2, x^\mu)$$

$$= \frac{m^2}{\lambda\beta}\Psi(\phi_1, \phi_2, x^\mu) \quad (21)$$

This equation describes a two-dimensional harmonic oscillator with frequency $\omega^2 = \frac{m_0^2 m^2}{\hbar^2 \lambda^2}$. By direct comparison with eq. (16) we obtain the following equation for the allowed energies (masses):

$$\frac{m^2}{\lambda\beta} = \hbar\sqrt{\frac{m_0^2 m^2}{\hbar^2\lambda^2}}(n + \frac{2}{2}), \quad n \in \mathcal{N} \tag{22}$$

with solution:

$$m = \pm\beta m_0(n+1), \quad n \in \mathcal{N} \tag{23}$$

Therefore, in addition to the positive 'masses' already found in Ref. [1], there is also a complete, similar spectrum of negative 'masses'.

III.III. The hydrogen atom

The Hamiltonian for the hydrogen atom is:

$$\mathcal{H} = \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{r}) \tag{24}$$

where $V(\mathbf{r})$ is the Kepler potential and μ is the reduced mass of the interacting particles.

It is easy to see that the stationary Schrödinger-like equation for this system is:

$$\left[\frac{-\hbar^2\nabla^2}{2\mu}\frac{E^2}{\lambda^2} + V(\mathbf{r})\right]\Psi(\mathbf{x}, t) = \frac{E^2}{\lambda\beta}\Psi(\mathbf{x}, t) \tag{25}$$

A direct comparison with the familiar expression for the standard hydrogen atom shows that the allowed energies, E_n, in eq. (25) have to be solutions of the equation:

$$\frac{E_n^2}{\lambda\beta} = \frac{\lambda^2}{E_n^2}\frac{E_{ground}}{n^2} \Rightarrow E_n^4 = \beta\lambda^3\frac{E_{ground}}{n^2} \tag{26}$$

It is clear that no choice of the parameters λ, β will reproduce the correct spectrum $E_n = \frac{E_{ground}}{n^2}$. Therefore, the new quantum theory *does not* reproduce the right spectrum for the hydrogen atom.

IV. More on the rules of interpretation

Even though it has been shown in the previous section that Good's proposal does not predict the correct experimental data, and is therefore not a valid quantum theory, it could be argued that *other* rules of quantization might lead to a good quantum theory based on the finite-dimensional covariant description in B1. In fact it is easy to show that the rules of interpretation in B1 are not equivalent to those in A1 and, therefore, they would not give rise to an equivalent theory regardless of the rules of interpretation they were supplemented with.

For the sake of simplicity we shall consider a space-time of the form $\{1, ..., N\} \times R$; that is to say, the space has only a finite- or, at most, countable -number of points.

In the description of the quantum theory in B1 the wave functions would be of the form

$$\Psi^B = \Psi^B(\varphi, n, t) \equiv \Psi_n^B(\varphi, t) \tag{27}$$

Thus, there is a function for each point of the space-time. The interpretation in B1 requires these wave functions to be square integrable:

$$\int \mathrm{d}\,\varphi |\Psi_n^B(\varphi, t)|^2 \in R \tag{28}$$

This must be compared with the standard description:

$$\Psi^A = \Psi^A(\varphi_1, ..., \varphi_N; t)\ . \tag{29}$$

The condition of square integrability here is:

$$\int \mathrm{d}\,\varphi_1, ...\mathrm{d}\,\varphi_N\, |\Psi^A(\varphi_1, ...\varphi_N; t)|^2 \in R \tag{30}$$

Now let us assume that the two functions $\Psi^A(\varphi_1, ..., \varphi_N)$ and $\Psi^B(\varphi, n)$ describe the same physical situation at $t = t_0$. Let us assume in addition that it is possible, at a fixed time, to measure the field φ at any point of the space without disturbing the measurements on the neighbouring points- this is always explicitly or implicitly assumed in the description in A1, whereas in the description in B1 it has to be considered as an additional postulate -. Then we would have:

$$|\Psi^A(\varphi_1, ..., \varphi_N)|^2 = |\Psi_1^B(\varphi_1)|^2 ... |\Psi_N^B(\varphi_N)|^2 \tag{31}$$

But it is evident that a general function $|\Psi^A(\varphi_1, ..., \varphi_N)|^2$ cannot always be decomposed as eq. (31) requires. Also, if two functions Ψ^A and Φ^A are decomposable, then the linear superposition of them, $\Psi^A + \Phi^A$, which, in the absence of superselection rules, is also an admissible function, will not generally admit such a decomposition.

Hence, there are physical situations which can be described with the standard quantum field theory but which do not admit a description within the formalism in B1.

V. Final comments

We have shown that the theory proposed in Ref. [1] is not equivalent

to the standard quantum theory either for mechanical systems- it does not reproduce, for instance, the right energy spectrum for the hydrogen atom -, or for field models- the physical interpretation is not the same -. Therefore, that formalism cannot describe the right 'physical' theory. Nonetheless, the proposal collects such a number of attractive properties that it is difficult not to believe that it must contain something true, perhaps as a limiting case of the standard theory. Therefore, it would be interesting to find a physical interpretation for it as well as for the 'mass' spectrum it 'predicts' for several fields.

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